

IMPLEMENTING THE KUSTIN-MILLER COMPLEX CONSTRUCTION

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ABSTRACT. The Kustin-Miller complex construction, due to A. Kustin and M. Miller, can be applied to a pair of resolutions of Gorenstein rings with certain properties to obtain a new Gorenstein ring and a resolution of it. It gives a tool to construct and analyze Gorenstein rings of high codimension. We describe the Kustin-Miller complex and its implementation in the Macaulay2 package KUSTINMILLER, and explain how it can be applied to explicit examples.

1. INTRODUCTION

Many important rings in commutative algebra and algebraic geometry turn out to be Gorenstein rings, i.e., commutative rings such that the localization at each prime ideal is a Noetherian local ring R with finite injective dimension as an R -module. Examples are canonical rings of regular algebraic surfaces of general type, anticanonical rings of Fano varieties and Stanley-Reisner rings of triangulations of spheres. Except for the complete intersection cases of codimension 1 and 2 a structure theorem for Gorenstein rings is known only for codimension 3 by the theorem of Buchsbaum-Eisenbud [5], which describes them in terms of Pfaffians of a skew-symmetric matrix. One goal of unprojection theory, which was introduced by A. Kustin, M. Miller and M. Reid and developed further by the second author (see, e.g., [8], [13], [12], [11]), is to act as a substitute for a structure theorem in codimension ≥ 4 by providing a construction to increase the codimension in a non-trivial way, while staying in the class of Gorenstein rings. The geometric motivation is to provide inverses of certain projections in birational geometry. The process can be considered as a version of Castelnuovo blow-down.

Examples of applications range from the construction of Campedelli surfaces [9] to results on the structure of Stanley-Reisner rings [2]. For an outline of more applications see [13], the introduction of [1] and Section 3 below.

We describe the Kustin-Miller complex construction [8], which is the key tool to obtain resolutions of unprojection rings, and discuss our implementation in the MACAULAY2 [7] package KUSTINMILLER [3]. We illustrate the construction with examples and applications.

2. IMPLEMENTATION OF THE KUSTIN-MILLER COMPLEX CONSTRUCTION

We will consider the following setup: Let R be a positively graded polynomial ring over a field and $I, J \subset R$ homogeneous ideals of R such that R/I and R/J are Gorenstein, $I \subset J$ and $\dim R/J = \dim R/I - 1$. By [4, Proposition 3.6.11] there are $k_1, k_2 \in \mathbb{Z}$ such that $\omega_{R/I} = R/I(k_1)$ and $\omega_{R/J} = R/J(k_2)$. Assume that $k_1 > k_2$ so that the unprojection ring defined below is also positively graded.

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Definition 1. [12] Let $\phi \in \text{Hom}_{R/I}(J, R/I)$ be a homomorphism of degree $k_1 - k_2$ such that $\text{Hom}_{R/I}(J, R/I)$ is generated as an R/I -module by ϕ and the inclusion morphism i . We call the graded algebra $R[T]/U$, where T is a variable of degree $k_1 - k_2$ and

$$U = (I, Tu - \phi(u) \mid u \in J)$$

the **Kustin–Miller unprojection ring** of the pair $I \subset J$ defined by ϕ .

Proposition 2. [8, 12] The ring $R[T]/U$ is Gorenstein and independent of the choice of ϕ (up to isomorphism).

Following [8], we now describe the construction of a graded free resolution of $R[T]/U$ from those of R/I and R/J . We will refer to this as the **Kustin–Miller complex construction**. Denote by $g = \dim R - \dim R/J$ the codimension of the ideal J of R , and suppose $g \geq 4$ (the special cases $g = 2$ and 3 can be treated in a similar way). Let

$$\begin{aligned} C_J : R/J &\leftarrow A_0 \xleftarrow{a_1} A_1 \xleftarrow{a_2} \dots \xleftarrow{a_{g-1}} A_{g-1} \xleftarrow{a_g} A_g \leftarrow 0 \\ C_I : R/I &\leftarrow B_0 \xleftarrow{b_1} B_1 \xleftarrow{b_2} \dots \xleftarrow{b_{g-1}} B_{g-1} \leftarrow 0 \end{aligned}$$

be minimal graded free resolutions (self-dual by the Gorenstein property, [6]) of R/J and R/I as R -modules with $A_0 = B_0 = R$, $A_g = R(k_1 - \eta)$ and $B_{g-1} = R(k_2 - \eta)$, where η is the sum of the degrees of the variables of R . Consider the complex

$$C_U : R[T]/U \leftarrow F_0 \xleftarrow{f_1} F_1 \xleftarrow{f_2} \dots \xleftarrow{f_{g-1}} F_{g-1} \xleftarrow{f_g} F_g \leftarrow 0$$

with the modules

$$\begin{aligned} F_0 &= B'_0, \quad F_1 = B'_1 \oplus A'_1(k_2 - k_1) \\ F_i &= B'_i \oplus A'_i(k_2 - k_1) \oplus B'_{i-1}(k_2 - k_1) \quad \text{for } 2 \leq i \leq g-2 \\ F_{g-1} &= A'_{g-1}(k_2 - k_1) \oplus B'_{g-2}(k_2 - k_1), \quad F_g = B'_{g-1}(k_2 - k_1) \end{aligned}$$

where for an R -module M we denote $M' := M \otimes_R R[T]$.

By specifying chain maps $\alpha : C_I \rightarrow C_J$, $\beta : C_J \rightarrow C_I[-1]$ and a homotopy map (not necessarily chain map) $h : C_I \rightarrow C_I$ we will define the differentials as

$$\begin{aligned} f_1 &= \begin{pmatrix} b_1 & \beta_1 + T \cdot a_1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} b_2 & \beta_2 & h_1 + T \cdot I_1 \\ 0 & -a_2 & -\alpha_1 \end{pmatrix} \\ f_i &= \begin{pmatrix} b_i & \beta_i & h_{i-1} + (-1)^i T \cdot I_{i-1} \\ 0 & -a_i & -\alpha_{i-1} \\ 0 & 0 & b_{i-1} \end{pmatrix} \quad \text{for } 3 \leq i \leq g-2 \\ f_{g-1} &= \begin{pmatrix} \beta_{g-1} & h_{g-2} + (-1)^{g-1} T \cdot I_{g-2} \\ -a_{g-1} & -\alpha_{g-2} \\ 0 & b_{g-2} \end{pmatrix}, \quad f_g = \begin{pmatrix} -\alpha_{g-1} + (-1)^g \frac{1}{\beta_g(1)} T \cdot a_g \\ b_{g-1} \end{pmatrix} \end{aligned}$$

where I_t denotes the rank $B_t \times \text{rank } B_t$ identity matrix. We now discuss the construction of α , β and h :

Fix R -module bases e_1, \dots, e_{t_1} of A_1 and $\hat{e}_1, \dots, \hat{e}_{t_1}$ of A_{g-1} and write

$$\sum_{i=1}^{t_1} \hat{c}_i \cdot \hat{e}_i := a_g(1_R), \quad c_i \cdot 1_R := a_1(e_i) \text{ for } i = 1, \dots, t_1$$

where by Gorensteinness $c_i, \hat{c}_i \in J$ for all i . Denote by $l_i, \hat{l}_i \in R$ lifts of $\phi(c_i), \phi(\hat{c}_i) \in R/I$, respectively. For an R -module A we write $A^* = \text{Hom}_R(A, R)$ and for an R -basis f_1, \dots, f_t of A we denote by f_1^*, \dots, f_t^* the dual basis of A^* . Now consider the R -homomorphism

$$A_{g-1}^* \rightarrow R = B_{g-1}^*, \quad \hat{e}_i^* \mapsto \hat{l}_i \cdot 1_R$$

which (by self-duality of C_I, C_J) extends to a chain map $C_J^* \rightarrow C_I^*$ and denote by $\tilde{\alpha} : C_I \rightarrow C_J$ its dual. The map $\tilde{\alpha}_0 : B_0 = R \rightarrow R = A_0$ is multiplication by an invertible element of R , cf. [11], set $\alpha = \tilde{\alpha}/\tilde{\alpha}_0(1_R)$.

We obtain $\beta : C_J \rightarrow C_I[-1]$ by extending

$$\beta_1 : A_1 \rightarrow R = B_0, \quad e_i \mapsto -l_i \cdot 1_R$$

Finally, by [8, p. 308] there is a homotopy $h : C_I \rightarrow C_I$ with $h_0 = h_{g-1} = 0$ and

$$\beta_i \alpha_i = h_{i-1} b_i + b_i h_i \quad \text{for } 1 \leq i \leq g$$

Theorem 3. [8] *The complex C_U is a graded free resolution of $R[T]/U$ as an $R[T]$ -module.*

It is important to remark that C_U is not necessarily minimal, although in many examples coming from algebraic geometry it is.

We now describe, how to compute C_U , as implemented in our MACAULAY2 package KUSTINMILLER. First note, that we can determine ϕ via the commands `Hom(J, R^1/I)` and `homomorphism` available in MACAULAY2. Furthermore one can extend homomorphisms to chain maps by the command `extend`.

Algorithm 1 Kustin-Miller complex

Input: Resolutions C_I and C_J , denoted as above, for homogeneous ideals $I \subset J$ in a polynomial ring R with R/I and R/J Gorenstein, and $\dim R/J = \dim R/I - 1$.

Output: The Kustin-Miller complex C_U associated to I and J .

- 1: Compute ϕ as in Definition 1 above.
- 2: Compute the dual C_J^* of C_J and express the first differential as the product of a square matrix Q with a_1 . Extend the homomorphism $\phi \circ Q$ to a chain map $\alpha^* : C_J^* \rightarrow C_I^*$ and dualize to obtain $\tilde{\alpha} : C_I \rightarrow C_J$. Dividing all differentials of $\tilde{\alpha}$ by the inverse of the entry of $\tilde{\alpha}_0$ yields $\alpha : C_I \rightarrow C_J$.
- 3: Extend the map $A_1 \rightarrow B_0$ given by ϕ to a chain map $C_J \rightarrow C_I[-1]$ and multiply the differentials by -1 to obtain $\beta : C_J \rightarrow C_I[-1]$.
- 4: Set $h_0 := 0_R$.
- 5: **for** $i = 1$ to $g - 1$ **do**
- 6: Set $h'_i := \beta_i \alpha_i - h_{i-1} b_i$.
- 7: Using the `extend` command obtain h_i in the diagram

$$\begin{array}{ccc} B_i & \xrightarrow{h'_i} & B_i \\ id \uparrow & & \uparrow b_i \\ B_i & \xrightarrow{h_i} & B_i \end{array}$$

8: **end for**

9: **return** the differentials f_i according to the formulas given above.

3. APPLICATIONS

We comment on some applications of the Kustin-Miller complex construction involving the authors (for examples on these, see the documentation of our MACAULAY2 package KUSTINMILLER).

3.1. Cyclic polytopes. For a polynomial ring $R = k[x_1, \dots, x_n]$ denote by $I_d(R)$ the Stanley-Reisner ideal of the boundary complex of the cyclic polytope of dimension d with vertices x_1, \dots, x_n . As shown in [2] the Kustin-Miller complex construction yields a recursion for a minimal resolution of $I_d(R)$: For d even apply Algorithm 1 with $T = x_n$ to minimal resolutions C_I and C_J of $I = I_d(k[x_1, \dots, x_{n-1}])$ and $J = I_{d-2}(k[z, x_2, \dots, x_{n-2}])$ considered as ideals in $k[z, x_1, \dots, x_{n-1}]$ and quotient by (z) . For d odd one can proceed in a similar way.

3.2. Stellar subdivisions. Suppose C is a Gorenstein* simplicial complex on the variables of $k[x_1, \dots, x_n]$ and F is a face of C . Let C_F be obtained by the stellar subdivision of C with respect to F , introducing the new variable x_{n+1} . Denote by I the image of the Stanley-Reisner ideal of C in $k[z, x_1, \dots, x_n]$ and by $J = (z) + I : (\prod_{i \in F} x_i)$ the ideal corresponding to the link of F . Apply Algorithm 1 to minimal resolutions of I and J with $T = x_{n+1}$ and quotient by (z) . By [1] this yields a resolution of the Stanley-Reisner ring of C_F .

3.3. Constructions in Algebraic Geometry. In the paper [9] a series of Kustin-Miller unprojections was used in order to give the first examples of Campedelli algebraic surfaces of general type with algebraic fundamental group $\mathbb{Z}/6$, while a similar technique produced in [10] seven families of Calabi-Yau 3-folds of high codimension. In both cases, the Kustin-Miller complex construction was used to control the numerical invariants of the new varieties.

4. EXAMPLE

Example 4. Using our MACAULAY2 package KUSTINMILLER [3] we discuss an example given in [11] passing from a codimension 3 to a codimension 4 ideal. Over the polynomial ring

```
i1: R = QQ[x_1..x_4, z_1..z_4];
```

consider the skew-symmetric matrix

```
i2: b2 = matrix{ { 0, x_1, x_2, x_3, x_4 },
                 { -x_1, 0, 0, z_1, z_2 },
                 { -x_2, 0, 0, z_3, z_4 },
                 { -x_3, -z_1, -z_3, 0, 0 },
                 { -x_4, -z_2, -z_4, 0, 0 } };
```

The Buchsbaum-Eisenbud complex

```
i3: betti( cI = resBE b2)
```

```
0 1 2 3
o3: total: 1 5 5 1
      0: 1 . . .
      1: . 5 5 .
      2: . . . 1
```

resolves the ideal $I = (b_1) \subset R$ generated by the 4×4 -Pfaffians

```
i4: b1 = cI.dd_1
```

```
o4: |z_2z_3-z_1z_4, -x_4z_3+x_3z_4, x_4z_1-x_3z_2, x_2z_2-x_1z_4, -x_2z_1+x_1z_3|
of the skew-symmetric matrix  $b_2$ . Consider the unprojection locus  $J$  with Koszul resolution
```

```
i5: J = ideal(z_1..z_4);
```

```
i6: betti( cJ = res J)
```

```
0 1 2 3 4
o6: total: 1 4 6 4 1
      0: 1 4 6 4 1
```

Applying Algorithm 1 we obtain the Kustin-Miller resolution of the unprojection ideal $U \subset R[T]$, in this case the ideal of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$,

```
i7: betti( cU = kustinMillerComplex(cI, cJ, QQ[T]))
```

```
0 1 2 3 4
o7: total: 1 9 16 9 1
      0: 1 . . . .
      1: . 9 16 9 .
      2: . . . . 1
```

with generators

i8: f1 = cU.dd_1

$$(b_1, -x_1x_3 + T \cdot z_1, -x_1x_4 + T \cdot z_2, -x_2x_3 + T \cdot z_3, -x_2x_4 + T \cdot z_4)$$

and syzygy matrix

i9: f2 = cU.dd_2

$$\left(\begin{array}{c|cccccc|ccccc} & 0 & 0 & 0 & 0 & 0 & 0 & T & 0 & 0 & 0 & 0 \\ & 0 & 0 & -x_1 & 0 & 0 & x_2 & 0 & T & 0 & 0 & 0 \\ b_2 & -x_1 & 0 & 0 & -x_2 & 0 & 0 & 0 & 0 & T & 0 & 0 \\ & 0 & 0 & -x_3 & -x_3 & -x_4 & 0 & -x_3 & 0 & 0 & T & 0 \\ & 0 & x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T \\ \hline & z_2 & z_3 & 0 & z_4 & 0 & 0 & z_4 & 0 & -x_4 & 0 & x_2 \\ 0 & -z_1 & 0 & z_3 & 0 & z_4 & 0 & 0 & 0 & x_3 & -x_2 & 0 \\ & 0 & -z_1 & -z_2 & 0 & 0 & z_4 & -z_2 & x_4 & 0 & 0 & -x_1 \\ & 0 & 0 & 0 & -z_1 & -z_2 & -z_3 & 0 & -x_3 & 0 & x_1 & 0 \end{array} \right)$$

The code computing this example and various others related to the applications mentioned above can be found in the documentation of the package KUSTINMILLER [3].

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